

The imprints of nonextensive statistical mechanics in high energy collisions*

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February 1, 2008

Abstract

To describe high energy collisions one widely uses thermodynamical methods and concepts which follow the classical Boltzmann-Gibbs (BG) approach. In many cases, however, either some deviations from the expected behaviour are observed experimentally or it is known that the conditions necessary for BG to apply are satisfied only approximately. In other branches of physics where such situations are ubiquitous, the popular remedy is to resort, instead, to the so called nonextensive statistics, the most popular example of which is Tsallis statistics. We shall provide here an overview of possible imprints of non-extensivity existing both in high energy cosmic ray physics and in multiparticle production processes in hadronic collisions, in particular in heavy ion collisions. Some novel proposition for the interpretation of the nonextensivity parameter q present in such circumstances will be discussed in more detail.

PACS numbers: 05.40.Fb 24.60.-k 05.10.Gg

Keywords: Nonextensive statistics, Lévy distributions, Thermal models

*Invited talk presented by G.Wilk at *International Workshop on Classical and Quantum Complexity and Non-extensive Thermodynamics* (Complexity 2000), Denton, University of North Texas, April 3-6, 2000.

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1 Introduction

It has been realized for some time already that in many places of science there are phenomena which clearly indicate the existence of some degree of nonextensivity (understood in the thermodynamical sense). They include all situations characterized by long-range interactions, long-range microscopic memory and space-time (and phase space as well) (multi)fractal structure of the process. Such anomalous (from the point of view of the standard thermodynamics and Boltzmann-Gibbs statistics) systems are found to be best described in terms of generalised, nonextensive thermostatics, the most popular and explored example of which is the so called Tsallis statistics which is characterised by the nonextensivity parameter q [1].

To make our presentation self-contained we shall first provide the basic formulas (referring to cf. [1] and references therein for a thorough discussion of all possible aspects of nonextensivity). Everything is based on a generalized entropic form depending on a single parameter (entropic index) q in such a way that for $q \rightarrow 1$ it gives the normal BG entropy:

$$S_q = -\frac{1}{1-q} \left(1 - \sum_i p_i^q \right) \xrightarrow{q \rightarrow 1} S_{BG} = -\sum_i p_i \ln p_i. \quad (1)$$

The S_q is nonextensive in the sense that

$$S_q(A+B) = S_q(A) + S_q(B) + (1-q) S_q(A)S_q(B), \quad (2)$$

where A and B are two independent systems in the usual sense, i.e., $p_{ij}(A+B) = p_i(A)p_j(B)$. In this sense the entropic index q is also a measure of the nonextensivity in the system. Using the usual procedure of information theory when looking for the most probable and least biased (normalized) probability distribution of some events x subjected to a single constraint in the form of unnormalized q -expectation value $\langle A \rangle_q = \int dx A(x) p(x)^q$ [2], one gets immediately, from the maximization of the entropy S_q the known expression

$$p_q(x) = \frac{1}{Z_q} [1 - (1-q)\alpha A(x)]^{\frac{1}{1-q}} \xrightarrow{q \rightarrow 1} \frac{1}{Z} \exp(-\alpha A(x)). \quad (3)$$

The Lagrange multiplier α can be expressed in terms of the $\langle A \rangle_q$ from the imposed constraint and Z_q from the normalization condition. Notice that whereas in the extensive, i.e., $q = 1$ case, all values of $x \in (0, \infty)$ are admissible, for nonextensive case of $q \neq 1$ we have restrictions so that $[1 - (1-q)\alpha A(x)]$ is positive.

Our presentation will be devoted to a very limited subject of high energy collisions. We shall attempt to overview the probable imprints of nonextensivity in high energy collisions where under this term we understand both multiparticle production processes taking place in cosmic ray experiments and those observed in accelerator experiments.

In cosmic ray experiments one encounters routinely a cascade processes, both in the atmosphere and in emulsion chambers serving as detectors. The former originate during the passage

of the primordial cosmic rays through the atmosphere with subsequent collisions and dissipation of energy (they are known as the so called Extensive Air Showers) - this is typical stochastic process not very much sensitive to details of the elementary interactions [3]. The latter are connected with the actual detection process taking place in special emulsion chambers exposed in many places of the Earth, usually at high altitudes. Some of their characteristics do depend on details of the interaction process. Because in both cases one encounters formulae of the type (3), they are *a priori* sensitive to the possible nonextensivity of such processes.

The accelerator high energy collisions are usually connected with production of large number of secondaries (mostly π and K mesons). The strong interactions involved here make their detail description *from first principles* impossible and one is forced to turn to phenomenological models of various kinds. The most economical (as far as the number of parameters is concerned) are thermodynamical and statistical models which have been, in fact, in use since the beginning of this field of research. It should be stressed that the very first successful phenomenological model of multiparticle production, the so called Landau Hydrodynamical Model, was proposed already in the pre-accelerator era of 1953 and used in the analysis of specific multiparticle data taken from cosmic ray interactions [4]. This model paved a way to more sophisticated statistical models now in use, mainly in high energy heavy ion collisions which are believed to lead to the production of new state of matter, the Quark Gluon Plasma (QGP) [6].

To recapitulate: in both cosmic ray and accelerator experiments one observes multiparticle production. However, in the cosmic ray case one also deals with propagation of the secondaries originating in multiple production processes at a given point and with their subsequent secondary interactions (also of multiparticle type), i.e., with a full fledged cascade process [3].

In Section 2 the possible traces of nonextensivity apparently seen in some cosmic ray experiments [7] with emulsion chambers will be presented along with similar effects seen in elementary e^+e^- [8] and hadronic collisions and in the collisions of nuclei in accelerators [9, 10]. In all of them distributions given by Eq. (3) are observed. Section 3 will be devoted to discussion on possible hints emerging from the occupation number distributions $\langle n \rangle_q$ [9, 10]. In Section 4 we shall propose a novel interpretation of the entropic index q [11] encountered in Sec. 2. The last Section contains final remarks together with a list of other possible hints of nonextensivity not discussed in detail here (including topics from QGP physics [12] and nonextensivity manifested in the statistics of the quantum states produced in the scattering process such as π -nucleon or π -nucleus scatterings [13]).

2 Traces of nonextensivity in nonexponential distributions

Our encounter with the notion of nonextensivity started when we realised that one of our previous results concerning the occurrence of the so called *long flying component* in the propagation of the initial flux of incoming cosmic ray particles (mostly nucleons) [14] can be interpreted as yet another manifestation of the Lévy distribution (3) [7]. To be more specific let us briefly summarize the result of [14]. We have analysed, there, the distribution of cascade starting points in the extrathick lead chamber of the Pamir experiment. The corresponding data points for the number of cascades originating at depth T (measured in cascade units, $1 \text{ c.u.} = 6.4 \text{ g/cm}^2 = 0.56 \text{ cm}$) are shown in Fig. 1. Whereas at small depths (up to $\sim 60 \text{ cm}$ of lead) we observe the usual absorption of hadrons as given by the simple exponential formula

$$\frac{dN}{dT} = \text{const} \cdot \exp\left(-\frac{T}{\lambda}\right), \quad (4)$$

at biggest thickness there is, against all expectations, noticeably excess of experimental points above the simple extrapolation of small-depth data. The observed discrepancy means that original hadrons tend to fly longer without interaction (that is why a term *long flying component* is coined for this type of phenomenon).

In [14] we have argued that the observed effect can be just another manifestation of the fluctuation of the corresponding hadronic cross section $\sigma = Am_N \frac{1}{\lambda}$ (where A denotes the mass number of the target and m_N is the mass of the nucleon, such a possibility is widely discussed in the literature and observed in diffraction dissociation experiments on accelerators, cf. [14] for details and references). It turned out that fluctuations of this cross section (i.e., in effect, fluctuations of the quantity $1/\lambda$) with relative variance

$$\omega = \frac{\langle \sigma^2 \rangle - \langle \sigma \rangle^2}{\langle \sigma \rangle^2} \geq 0.2 \quad (5)$$

allow to describe the observed effect.

It turns out [7] that the same data can be fitted by the nonextensive formula

$$\frac{dN}{dT} = \text{const} \cdot \left[1 - (1-q) \frac{T}{\lambda} \right]^{\frac{1}{1-q}} \quad (6)$$

with parameter $q = 1.3$ (in both cases $\lambda = 18.85 \pm 0.66$ in *c.u.* defined above), cf. Fig. 1 [15].

Similar example is also known in heavy ion collisions [10] (see also [9]). It turns out that distributions of transverse momenta of secondaries produced in nuclear collisions at high energies (transverse with respect to the collision axis given by the direction of the colliding objects in the center of mass frame) $dN(p_T)/dp_T$ are best described by a slightly nonexponential function

of the type

$$\frac{dN(p_T)}{dp_T} = \text{const} \cdot \left[1 - (1 - q) \frac{\sqrt{m^2 + p_T^2}}{kT} \right]^{\frac{1}{1-q}} \xrightarrow{q \rightarrow 1} \text{const} \cdot \exp \left(-\frac{\sqrt{m^2 + p_T^2}}{kT} \right). \quad (7)$$

Here m is the mass of produced particle, k is the Boltzmann constant (which we shall, in what follows, put equal unity) and T is, for the $q = 1$ case, the *temperature* of the reaction considered (or, rather, the temperature of the hadronic system produced). In fact, precisely from such exponential fits to transverse masses $m_T = \sqrt{m^2 + p_T^2}$ one infers information about the temperature T . Therefore, any deviation from the exponential behaviour of such distributions are always under detailed scrutiny in which one is searching for the possible causes. In [9] it was suggested that the extreme conditions of high density and temperature occurring in ultra-relativistic heavy ion collisions can lead to memory effects and long-range colour interactions and to the presence of non-Markovian processes in the corresponding kinetic equations [16].

As it is seen in Fig. 2 one indeed finds [9, 10] a small deviation from the exponential behaviour (on the level $q = 1.015$). As we shall demonstrate in Section 4 it can, however, lead to quite dramatic effects. It was also shown in [9] that to first order in $|q - 1|$ the generalized slope becomes the quantity

$$T_q = T + (q - 1)m_T. \quad (8)$$

with T being temperature of a *purely thermal source*. This should be contrasted with the empirical relation for the slope parameter T , from which the freeze-out temperature (at which hadrons are created from the QGP) T_f is then deduced,

$$T = T_f + m\langle v_\perp \rangle^2. \quad (9)$$

The $\langle v_\perp \rangle$ is a fit parameter usually identified with the average collective (transverse) flow velocity of the hadrons being produced. In (8) one has, instead, a *purely thermal source* experiencing a kind of blue shift at high m_T (actually increasing with m_T). The nonextensivity parameter q accounts here for all possibilities one can find in [16] and could, therefore, be regarded as a new way of presenting experimental results with $q \neq 1$ signaling that there is something going on in the collision that prevents it to from being exactly thermal-like in the ordinary sense mentioned above.

We shall proceed, now, to two other examples of possible nonexponential distributions. First is the attempt [8] to fit the energy spectra in both the longitudinal and transverse momenta of particles produced in the e^+e^- annihilation processes at high energies. Those are, contrary to the previous example, the most elementary high energy multiparticle production processes. The initial e^+e^- pair annihilates to a virtual photon which subsequently gives rise to a (highly excited) quark-antiquark pair. They in turn develop a complex hadronization process related to the long-distance (strong coupling) regime of Quantum Chromodynamic (QCD). Usually being described in terms of the so called string model [17] it admits also, for low energies, a

kind of thermodynamical equilibrium approach [8]. However, it turns out that when going to higher energies one cannot keep the temperature T_0 inferred from the p_T distributions constant, invalidating therefore the whole concept. On the other hand, using instead the nonextensive power-like (q -dependent) distribution, one can write the following transverse momentum distribution [8]:

$$\frac{1}{\sigma} \frac{d\sigma}{dp_T} = \text{const} \cdot p_T \int_0^\infty dp_L \left[1 - (1-q) \frac{\sqrt{p_L^2 + m^2 + p_T^2}}{T_0} \right]^{\frac{1}{1-q}} \quad (10)$$

(where p_L is longitudinal momentum of secondary particle of mass m and q is the entropic index). Keeping the temperature T_0 essentially constant and changing only q one can now fit data extremely well, cf. Fig. 3 [8]. In this example the $q \neq 1$ is then regarded as a manifestation of nonextensivity arising in the hadronization process in which quarks and gluons combine together forming hadrons, a process which involves long range correlations in the phase space. Actually, this observation has general validity and applies to all production processes discussed here as well. It applies, most probably, also to a pure elastic quantum scattering processes discussed in [13].

The final example from this category deals with the most probable rapidity distributions for produced secondaries. Suppose that we have an excited object of mass M which hadronizes into N secondaries of transverse mass m_T each. For simplicity we shall consider only one-dimensional hadronization in which transverse momenta are hidden in the mean $\langle p_T \rangle$ parameter, which is kept constant and enters m_T . In this case the observable distribution of interest is $f(y) = \frac{1}{N} \frac{dN}{dy}$, i.e., a distribution in rapidity defined as $y = \frac{1}{2} \ln[(E + p_L)/(E - p_L)]$. Some time ago it was shown, using the maximization of the information entropy in the Shannon form (i.e., in fact the BG one), that knowing only the energy M and multiplicity N one has to expect that [18]

$$f(y) = \frac{1}{Z(M, N)} \exp[-\beta(M, N) \cdot m_T \cosh y], \quad (11)$$

where $Z(M, N)$ comes from the normalization of $f(y)$ and the Lagrange multiplier $\beta(M, N)$ from the energy conservation constraint. Of special interest to us is the fact that for some values of the mean energy M/N (i.e., for some values of N for a given mass M) β can be zero and even negative. Actually it can be shown that $\beta \geq 0$ only if

$$N \geq N_0 \simeq 2 \ln \frac{M}{m_T}. \quad (12)$$

This statement invalidates the widely assumed, on different occasions, the so-called *Feynman scaling* hypothesis, namely, that in high energy collisions one should expect $f(y) \simeq \text{const}$ (cf. [18] for more information). Such hypothesis is clearly incompatible with the experimentally observed fact that the mean multiplicity $\langle N \rangle$ grows with energy faster than $\ln M$, more like $\langle N \rangle \sim M^{0.4-0.5}$). However, using Tsallis q -entropy, instead of BG one obtains [19]

$$f_q(y) = \frac{1}{Z_q(M, N)} [1 - (1-q)\beta_q(M, N) \cdot m_T \cosh y]^{\frac{1}{1-q}} \quad (13)$$

with $\beta_q(M, N) \geq 0$ for

$$N \geq N_{q0} \simeq 2 \left(\ln \frac{M}{m_T} \right)^q. \quad (14)$$

It is, then, obvious that for $q > 1$ one can indeed accommodate, at a given rapidity interval, more particles with $f(y) = f_q(y)$ being constant. In a sense one can think of a kind of *Feynman q -scaling* here (which would generalize the usual one and describe, in terms of the parameter q , what is usually called a violation of the Feynman scaling hypothesis [20]). In Fig. 4 this effect is clearly demonstrated for $M = 100$ GeV (and $m_T = 0.4$ GeV). While in Fig. 4a $N = 10$ is chosen in such a way as to have $\beta(100, 10) \simeq 0$ (notice that in this case $\beta_{q=0.7} > 0$ whereas $\beta_{q=1.3} < 0$), in Fig. 4b one has, instead, $\beta_{q=1.3}(100, 20) = 0$ (and remaining $\beta_{q=1} > 0$ and $\beta_{q=0.7} > 0$). Fig 4c shows, for comparison, the case with $N > N_0$ for which all $\beta_q > 1$.

3 Traces of nonextensivity in the mean occupation numbers n_q

Another place where nonextensivity enters in a natural way is the mean occupation numbers generalizing the Bose-Einstein or Fermi-Dirac ones to a $q \neq 1$ case. Whereas the single particle distribution function is obtained in the usual procedure of maximizing the Tsallis entropy under the constraints of given average internal energy and number of particles, the mean occupation numbers $\langle n \rangle_q$ are not available in analytical formula for any q . Only in the dilute gas approximation and for small deviations of q from unity can one express them in a simple analytical form [21]

$$\langle n \rangle_q = \left\{ [1 + (q-1)\beta(E-\mu)]^{1/(q-1)} \pm 1 \right\}^{-1}, \quad (15)$$

where $\beta = 1/kT$, μ is the chemical potential and the $+/-$ sign applies to fermions/bosons. Notice that in the limit $q \rightarrow 1$ (extensive statistics) one recovers the conventional Fermi-Dirac and Bose-Einstein distributions. What will interest us here are the generalized particle fluctuations,

$$\langle \Delta n^2 \rangle_q \equiv \frac{1}{\beta} \frac{\partial \langle n \rangle_q}{\partial \mu} = \frac{\langle n \rangle_q}{1 + (q-1)\beta(E-\mu)} (1 \mp \langle n \rangle_q), \quad (16)$$

where $E = \sqrt{m^2 + p^2}$. That is because, so far, these formulas have been applied to study the fluctuation pattern expected in heavy ion collisions [9, 10] in measurements performed on an event-by-event basis. Notice that the denominator occurring in (16) modulates in a novel and specific way the usual pattern of fluctuations for the $q = 1$ case [9].

Event-by-event fluctuations can be used as a valuable source of information on the dynamics of heavy-ion collisions. However, there is the problem of how to disentangle the *dynamical* fluctuations of interest from the *trivial* geometrical ones due to the impact parameter variation (resulting in the different number of nucleons participating in a given event). To solve this

problem it was proposed in [22] to use the following measure of fluctuations or correlations:

$$\Phi_x = \sqrt{\frac{\langle Z^2 \rangle}{\langle N \rangle}} - \sqrt{z^2} \quad \text{where} \quad Z = \sum_{i=1}^N z_i. \quad (17)$$

Here $z_i = x_i - \bar{x}$ where \bar{x} denotes the mean value of the observable x calculated for all particles from all events (the so called inclusive mean) and N is the number of particles analysed in the event. In (17) $\langle N \rangle$ and $\langle Z^2 \rangle$ are averages of event-by-event observables over all events whereas the last term is the square root of the second moment of the inclusive z distribution. Φ equals zero when the correlations are entirely absent. On the other hand, it is constructed in such a way as to be exactly the same for nucleon-nucleon and nucleus-nucleus collisions if the latter is a simple superposition of the former. The Φ -measure has been successfully applied to the experimental data (cf. [23]) and the fluctuations of transverse momentum p_T , which are observed in nucleon-nucleon collisions, have been found to be significantly reduced in the central Pb-Pb collisions at 158 GeV per nucleon. Its nonextensive extension has been analysed in [9] where it was found that the usual correlations are increased for $q < 1$ and decreased for $q > 1$. This time the measure Φ can become positive or negative for both fermions and bosons depending on the value of the entropic index q . Actually in [9] it was found that for $T = 140$ MeV and $\mu = 0$ the Φ -measure vanishes for $q = 1.015$.

The clear prediction of [9], which can consist a subject for experimental verification, is that the p_T -dependence of partial contributions to Φ_{p_T} should become negative for $p_T > 0.5$ GeV. Notice that in this regime the Φ -measure should already be free from contaminations from resonance decays and, therefore, the experimental confirmation of this prediction would provide a strong signal for the nonextensivity present in relativistic heavy ion collisions.

The Φ -measure is applicable not only to fluctuations of kinematical quantities such as p_T but also to the azimuthal [24] and chemical fluctuations as well. The latter were analysed in [25] for the normal statistics and in [10] for the nonextensive one. The representative sample of results is shown in Fig. 5 for $q = 1.015$ mentioned above. For simplicity we have restricted ourselves, here, only to comparison with results of [25] without resonances [26]. Actually, one expects that for a given q fluctuations should grow with the mass of detected particle - this observation provides yet another possibility for experimental verification of the nonextensivity concept.

4 Nonextensivity parameter q as a measure of fluctuations

The general picture emerging from the previous discussion is that one can account very economically (by introducing only one new parameter q) and adequately (by using nonextensive formulas emerging from Tsallis statistics with entropic index q) for a number of observations

deviating from the normal BG approach. The question of the possible meaning of entropic index in these cases is therefore very natural. For the cases discussed in Section 2 we would like to propose that q is connected with fluctuations present in the system under investigation. Notice that common feature of the first two examples in Section 2 is that they both are described by the powerlike distribution of the type

$$L_q(\varepsilon) = C_q \left[1 - (1 - q) \frac{\varepsilon}{\lambda} \right]^{\frac{1}{1-q}}. \quad (18)$$

As mentioned before, the cosmic ray example was originally explained [14] by the apparent fluctuation of the mean free path parameter λ in the corresponding exponential formula (4). It is then natural to expect that these fluctuations (which were so far described in [14] only numerically by means of Monte Carlo simulations) should be formulated in such a way as to result in eq. (18) with a parameter q . The same should be also true for the heavy ion collision example. Actually, this example is even more important and interesting because of the long and still vivid discussion on the possible dynamics of temperature fluctuations [27, 28, 29, 30] and because of its connection with the problem of QGP production in heavy ion collisions [28, 30]. We shall, therefore, treat both cases as representing the same class of fluctuation phenomena and claim that *the parameter q is a measure of fluctuations* present in the Lévy-type distributions (18) describing the particular process under consideration.

To demonstrate this conjecture let us analyse the influence of fluctuations of the parameter $1/\lambda$ present in the exponential formula $L_{q=1}(\varepsilon) \sim \exp(-\varepsilon/\lambda)$. Our aim will be to deduce the form of a function $f(1/\lambda)$ which transforms the exponential distribution to a power-like Lévy distribution (18) and which describes fluctuations about the mean value $1/\lambda_0$. Although in both examples considered above the data preferred $q > 1$, we shall discuss $q < 1$ case as well. In the $q > 1$ case, where $\varepsilon \in (0, \infty)$, one has,

$$L_{q>1}(\varepsilon; \lambda_0) = C_q \left(1 + \frac{\varepsilon}{\lambda_0} \frac{1}{\alpha} \right)^{-a} = C_q \int_0^\infty \exp\left(-\frac{\varepsilon}{\lambda}\right) f\left(\frac{1}{\lambda}\right) d\left(\frac{1}{\lambda}\right) \quad (19)$$

where $\alpha = \frac{1}{q-1}$. Writing the following representation of the Euler gamma function [31],

$$\left(1 + \frac{\varepsilon}{\lambda_0} \frac{1}{\alpha} \right)^{-a} = \frac{1}{\Gamma(\alpha)} \int_0^\infty d\eta \eta^{\alpha-1} \exp\left[-\eta \left(1 + \frac{\varepsilon}{\lambda_0} \frac{1}{\alpha} \right)\right], \quad (20)$$

and changing variables under the integral to $\eta = \alpha \frac{\lambda_0}{\lambda}$, one obtains eq. (19) with $f(1/\lambda)$ given by the following gamma distribution:

$$f_{q>1}\left(\frac{1}{\lambda}\right) = f_\alpha\left(\frac{1}{\lambda}, \frac{1}{\lambda_0}\right) = \frac{\mu}{\Gamma(\alpha)} \left(\frac{\mu}{\lambda}\right)^{\alpha-1} \exp\left(-\frac{\mu}{\lambda}\right) \quad (21)$$

with $\mu = \alpha \lambda_0$ and with mean value and variation in the form:

$$\left\langle \frac{1}{\lambda} \right\rangle = \frac{1}{\lambda_0} \quad \text{and} \quad \left\langle \left(\frac{1}{\lambda} \right)^2 \right\rangle - \left\langle \frac{1}{\lambda} \right\rangle^2 = \frac{1}{\alpha \lambda_0^2}. \quad (22)$$

Notice that, with increasing α the variance (22) decreases and asymptotically (for $\alpha \rightarrow \infty$, i.e., for $q \rightarrow 1$) the gamma distribution (21) becomes a delta function, $f_{q>1}(1/\lambda) = \delta(\lambda - \lambda_0)$. The relative variance for this distribution is given by

$$\omega = \frac{\left\langle \left(\frac{1}{\lambda}\right)^2 \right\rangle - \left\langle \frac{1}{\lambda} \right\rangle^2}{\left\langle \frac{1}{\lambda} \right\rangle^2} = \frac{1}{\alpha} = q - 1. \quad (23)$$

For the $q < 1$ case ε is limited to $\varepsilon \in [0, \lambda_0/(1 - q)]$. Proceeding in the same way as before, i.e., making use of the following representation of the Euler gamma function (where $\alpha' = -\alpha = \frac{1}{1-q}$)

$$\left[1 - \frac{\varepsilon}{\alpha' \lambda_0}\right]^{\alpha'} = \left(\frac{\alpha' \lambda_0}{\alpha' \lambda_0 - \varepsilon}\right)^{-\alpha'} = \frac{1}{\Gamma(\alpha')} \int_0^\infty d\eta \eta^{\alpha'-1} \exp\left[-\eta \left(1 + \frac{\varepsilon}{\alpha' \lambda_0 - \varepsilon}\right)\right], \quad (24)$$

and changing variables under the integral to $\eta = \frac{\alpha' \lambda_0 - \varepsilon}{\lambda}$, we obtain $L_{q<1}(\varepsilon; \lambda_0)$ in the form of eq. (19) but with $\alpha \rightarrow -\alpha'$ and with the respective $f(1/\lambda) = f_{q<1}(1/\lambda)$ given now by the same gamma distribution as in (21) but this time with $\alpha \rightarrow \alpha'$ and $\mu = \mu(\varepsilon) = \alpha' \lambda_0 - \varepsilon$. Contrary to the $q > 1$ case, this time the fluctuations depend on the value of the variable in question, i.e., the mean value and variance are now both ε -dependent:

$$\left\langle \frac{1}{\lambda} \right\rangle = \frac{1}{\lambda_0 - \frac{\varepsilon}{\alpha'}} \quad \text{and} \quad \left\langle \left(\frac{1}{\lambda}\right)^2 \right\rangle - \left\langle \frac{1}{\lambda} \right\rangle^2 = \frac{1}{\alpha'} \cdot \frac{1}{\left(\lambda_0 - \frac{\varepsilon}{\alpha'}\right)^2}. \quad (25)$$

However, the relative variance

$$\omega = \frac{\left\langle \left(\frac{1}{\lambda}\right)^2 \right\rangle - \left\langle \frac{1}{\lambda} \right\rangle^2}{\left\langle \frac{1}{\lambda} \right\rangle^2} = \frac{1}{\alpha'} = 1 - q, \quad (26)$$

remains ε -independent and depends only on the parameter q . As above the resulting gamma distribution becomes a delta function, $f_{q<1}(1/\lambda) = \delta(\lambda - \lambda_0)$, for $\alpha' \rightarrow \infty$, i.e., for $q \rightarrow 1$.

This completes the proof of our conjecture. The nonextensivity parameter q in the $L_q(\varepsilon)$ distributions can, indeed, be expressed by the relative variance ω of fluctuations of the parameter $1/\lambda$ in the distribution $L_{q=1}(\varepsilon)$:

$$q = 1 \pm \omega \quad (27)$$

for the $q > 1$ (+) and $q < 1$ (−) cases.

Concerning transverse momentum distributions in heavy ion collisions, $dN(p_T)/dp_T$, it is interesting to notice that the relatively small value $q \simeq 1.015$ of the nonextensive parameter obtained there [9, 10], if interpreted in the same spirit as above, indicates that rather large relative fluctuations of temperature, of the order of $\Delta T/T \simeq 0.12$, exist in nuclear collisions. It could mean therefore that we are dealing here with some fluctuations existing in small parts of the system in respect to the whole system (according to interpretation of [27]) rather than

with fluctuations of the event-by-event type in which, for large multiplicity N , fluctuations $\Delta T/T = 0.06/\sqrt{N}$ should be negligibly small [30]. This controversy could be, in principle, settled by detailed analyses of the event-by-event type. Already at present energies and nuclear targets (and the more so at the new accelerators for heavy ions like RHIC at Brookhaven, now commissioned, and LHC at CERN scheduled to be operational in the year 2006) one should be able to check whether the power-like p_T distribution $dN(p_T)/dp_T$ occurs already at every event or only after averaging over all events. In the former case we would have a clear signal of thermal fluctuations of the type mentioned above. In the latter case one would have for each event a fixed T value which would fluctuate from one event to another (most probably because different initial conditions are encountered in a given event).

The proposed interpretation of q leads immediately to the next question: why and under what circumstances is it the gamma distribution that describes fluctuations of the parameter λ ? To address it let us write the usual Langevin equation for the stochastic variable λ [32]:

$$\frac{d\lambda}{dt} + \left[\frac{1}{\tau} + \xi(t) \right] \lambda = \phi = \text{const} > 0. \quad (28)$$

with damping constant τ and source term ϕ . This term will be different for the two cases considered, namely:

$$\phi = \phi_{q<1} = \frac{1}{\tau} \left(\chi_0 - \frac{\varepsilon}{\alpha'} \right) \quad \text{whereas} \quad \phi = \phi_{q>1} = \frac{\chi_0}{\tau}. \quad (29)$$

For stochastic processes defined by the *white gaussian noise* form of $\xi(t)$ [33] one obtains the following Fokker-Plank equation for the distribution function of the variable λ [34]

$$\frac{df(\lambda)}{dt} = - \frac{\partial}{\partial \lambda} K_1 f(\lambda) + \frac{1}{2} \frac{\partial^2}{\partial \lambda^2} K_2 f(\lambda), \quad (30)$$

where the intensity coefficients $K_{1,2}$ are defined by eq.(28) and are equal to (cf., for example, [35]):

$$K_1(\lambda) = \phi - \frac{\lambda}{\tau} + D \lambda \quad \text{and} \quad K_2(\lambda) = 2 D \lambda^2. \quad (31)$$

From it we get the following expression for the distribution function of the variable λ :

$$f(\lambda) = \frac{c}{K_2(\lambda)} \exp \left[2 \int_0^\lambda d\lambda' \frac{K_1(\lambda')}{K_2(\lambda')} \right] \quad (32)$$

which is, indeed, a gamma distribution in variable $1/\lambda$,

$$f(\lambda) = \frac{1}{\Gamma(\alpha)} \mu \left(\frac{\mu}{\lambda} \right)^{\alpha-1} \exp \left(- \frac{\mu}{\lambda} \right), \quad (33)$$

with the constant c defined by the normalization condition, $\int_0^\infty d(1/\lambda) f(1/\lambda) = 1$ and depending on two parameters:

$$\mu(\varepsilon) = \frac{\phi_q(\varepsilon)}{D} \quad \text{and} \quad \alpha_q = \frac{1}{\tau D}, \quad (34)$$

with $\phi_q = \phi_{q>1, q<1}$ and $\alpha_q = (\alpha, \alpha')$ for, respectively, $q > 1$ and $q < 1$. This means that we have obtained eq. (27) with $\omega = \frac{1}{\tau D}$ and, therefore, the parameter of nonextensivity q is given

by the parameter D and by the damping constant τ describing the *white noise*.

The above discussion rests on the stochastic equation (28). To comment on its possible origin let us turn once more to fluctuations of temperature [27, 28, 29, 30] discussed before, i.e., to $\lambda = T$. Suppose that we have a thermodynamic system, in a small (mentally separated) part of which the temperature fluctuates with $\Delta T \sim T$. Let $\lambda(t)$ describe stochastic changes of the temperature in time. If the mean temperature of the system is $\langle T \rangle = T_0$ then, as result of fluctuations in some small selected region, the actual temperature equals $T' = T_0 - \tau \xi(t) T$. The inevitable exchange of heat between this selected region and the rest of the system leads to the equilibration of the temperature and this process is described by the following equation [36]

$$\frac{\partial T}{\partial t} - \frac{1}{\tau} (T' - T) + \Omega_q = 0 \quad (35)$$

which is, indeed, of the type of eq. (28) (here $\Omega_{q<1} = \frac{\varepsilon}{\tau \alpha'}$ and $\Omega_{q>1} = 0$).

In this way we have recovered eq. (28) and clearly demonstrated the plausibility of our proposition. Notice the presence of the internal heat source in the above equation in the $q < 1$ case. It has a sense of dissipative transfer of energy from the region where (due to fluctuations) the temperature T is higher. It could be any kind of convection-type flow of energy; for example, it could be connected with emission of particles from that region. The heat release given by $\varepsilon/(\tau \alpha')$ depends on ε (but it is only a part of ε that is released). In the case of such energy release (connected with emission of particles) there is additional cooling of the whole system. If this process is sufficiently fast, it could happen that there is no way to reach a stationary distribution of temperature (because the transfer of heat from the outside can be not sufficient for the development of the state of equilibrium). On the other hand (albeit this is not our case here) for the reverse process we could face the "heat explosion" situation (which could happen if the velocity of the exothermic burning reaction grows sufficiently fast; in this case because of nonexistence of stationary distribution we have fast nonstationary heating of the substance and acceleration of the respective reaction).

It should be noticed that in the case of $q < 1$ the temperature does not reach stationary state because, cf. Eq. (25), $\langle 1/T \rangle = 1/(T_0 - \varepsilon/\alpha')$, whereas for $q > 1$ we had $\langle 1/T \rangle = 1/T_0$. As a consequence the corresponding Lévy distributions are defined only for $\varepsilon \in (0, T_0 \alpha')$ because for $\varepsilon \rightarrow T_0 \alpha'$, $\langle T \rangle \rightarrow 0$. Such asymptotic (i.e., for $t/\tau \rightarrow \infty$) cooling of the system ($T \rightarrow 0$) can be also deduced from Eq. (35) for $\varepsilon \rightarrow T_0 \alpha'$.

Our explanation, being tied to specific examples (especially to the example of the temperature fluctuations) differs from other works in which $L_{q \neq 1}(\varepsilon)$ is shown to be connected with $L_{q=1}(\varepsilon)$ by the so called Hilhorst integral formula (the trace of which is our eq. (20)) [31, 37] but without discussing the physical context of the problem. Our original motivation was to understand the apparent success of Tsallis statistics (i.e., the situations in which $q > 1$ or,

possibly also $q < 1$) in the realm of high energy collisions. It should be stressed that in this way we have addressed the interpretation of only very limited cases of applications of Tsallis statistics. They belong to the category in which the power laws physically appear as a consequence of some continuous spectra within appropriate integrals. It does not touch, however, a really *hard* case of applicability of Tsallis statistics, namely when *zero* Lyapunov exponents are involved [38]. Nevertheless, this allows us to interpret some nuclear collisions data in terms of fluctuations of the inverse temperature, providing thus an important hint to the origin of some systematics in the data, understanding of which is crucial in the search for a new state of matter, the Quark Gluon Plasma [9, 30].

5 Final remarks

There are also other imprints of nonextensivity which we shall only mention. One is connected with recent analysis [12] of the equilibrium distribution of heavy quarks in Fokker-Planck dynamics. It was demonstrated that thermalization of charmed quarks in a QGP proceeding via collisions with light quarks and gluons results in a spectral shape which can be described only by the Tsallis distribution [39]. On the other hand in [13] the quantum scattering processes (such as πN and πA) scatterings were analysed using Tsallis-like entropies and strong evidence for the nonextensivity were found there when analysing the experimental data on the respective phase shifts. On the boundary of really high energy collisions is the very recent application of the nonextensive statistics to the nuclear multifragmentation processes [40]. The other examples do not refer to Tsallis thermostatistics directly, nevertheless it can be demonstrated that they are, at least approximately, connected to it. We would like to refer here to a recent attempt to study, by using the formalism of quantum groups, the so called Bose-Einstein correlations between identical particles observed in multiparticle reactions [41] and also works on intermittency and multiparticle distributions using the so called Lévy stable distributions [42]. They belong, in some sense, to the domain of nonextensivity because, as was shown in [43], there is close correspondence between the deformation parameter of quantum groups used in [41] and the nonextensivity parameter q of Tsallis statistics and there is also connection between Tsallis statistics and Lévy stable distributions [44]. Some traces of the possible nonextensive evolution of cascade type hadronization processes were also searched for in [45]. The quantum group approach [41, 43] could probably be a useful tool when studying delicate problem of interplay between QGP and hadrons produced from it. It is plausible that description in terms of q -deformed bosons (or the use of some kind of interpolating statistics) would lead to more general results than the simple use of nonextensive mean occupation numbers $\langle n \rangle_q$ discussed above (for which the only known practical description is limited to small deviations from nonextensivity only).

To the extend to which self-organized criticality (SOC) is connected with nonextensivity [1] one should also mention here a very innovative (from the point of view of high energy collision)

application of the concept of SOC to such processes [46].

To summarize, it has been demonstrated that multiparticle processes bear also some signs of nonextensivity observed in other branches of physics, which shows up only as small deviations from the expected behaviour. These deviations were already explained by invoking some additional mechanisms and, because of this, the use of q -statistics is not so popular or known in this field as in others discussed in [1]. The advantage of the use of q -statistics is probably best seen from the information theoretical point of view. The new parameter q can be regarded then as a kind of compactification of all processes responsible for the actual nonextensivity into one single number [47]. This is also the point of view expressed in [48] where the new approach to quantum field theory based on Lorentzian, instead of Gaussian, path integrals has been proposed. It would allow to account for the possible deviations of pure stochasticity in a similarly most economical way when one introduces a single new parameter. This is, however, so far unexplored domain of research.

Acknowledgements: The partial support of Polish Committee for Scientific Research (grant 621/E-78/SPUB/CERN/P-03/DZ4/99) is acknowledged. One of us (G.W.) is very grateful to Prof. P.Grigoletti and Prof. C.Tsallis, organizers of the Complexity 2000 Workshop, for the financial support and warm hospitality extended to him during the conference. We thank Dr. A.Petridis for critical reading of the manuscript.

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Figure Captions

- Fig. 1** Depth distribution of the starting points, $dN(T)/dT$, of cascades in Pamir lead chamber. Notice the non-exponential behaviour of data points (for their origin cf. [14]) which can be fitted by Tsallis distribution (6) with $q = 1.3$. (This figure is reproduced from Fig. 1 of [7]).
- Fig. 2** The results for p_T distribution $dN(p_T)/dp_T$: notice that $q = 1.015$ results describes also the tail of distribution not fitted by the conventional exponent (i.e., $q = 1$). This figure is reproduced from Fig. 3 of [10].
- Fig. 3** The example (Fig. 1 of [8]) of transverse momentum distributions for e^+e^- annihilation processes for different energies (shown in Figure). The inset shows region of small values of p_T . The dotted line shows curve of constant T and $q = 1$. Other curves were obtained by fitting corresponding data with essentially fixed $T \simeq 110$ MeV and q growing fast with energy to stabilize at ~ 90 GeV at value $q = 1.2$ (cf. [8] for details; this figure is reproduced from Fig. 1 of [8]).
- Fig. 4** The examples of the most probable rapidity distributions obtained by extending analysis of [18] (eq. (11)) to the nonexponential ($q \neq 1$) distributions given by eq. (13). The object (fireball, string,...) of mass $M = 100$ GeV decays into N secondaries of (transverse) mass $m_T = 0.4$ GeV each. Figs. (a) and (b) show results for N leading to *Feynman scaling* ($\beta = 0$) or *Feynman q -scaling* ($\beta_{q=1.3} = 0$). Fig. (c) shows example of such N that all $\beta > 0$.
- Fig. 5** Example of the nonextensivity in fluctuations: Fig. 1 of [10] showing Φ - measure of the kaon multiplicity fluctuations (in the π^-K^- system of particles) as a function of temperature for three values of the pion chemical potential. The kaon chemical potential vanishes. The resonances are neglected. (a) - results of [25] (in linear scale); (b) - our results for $q = 1.015$.

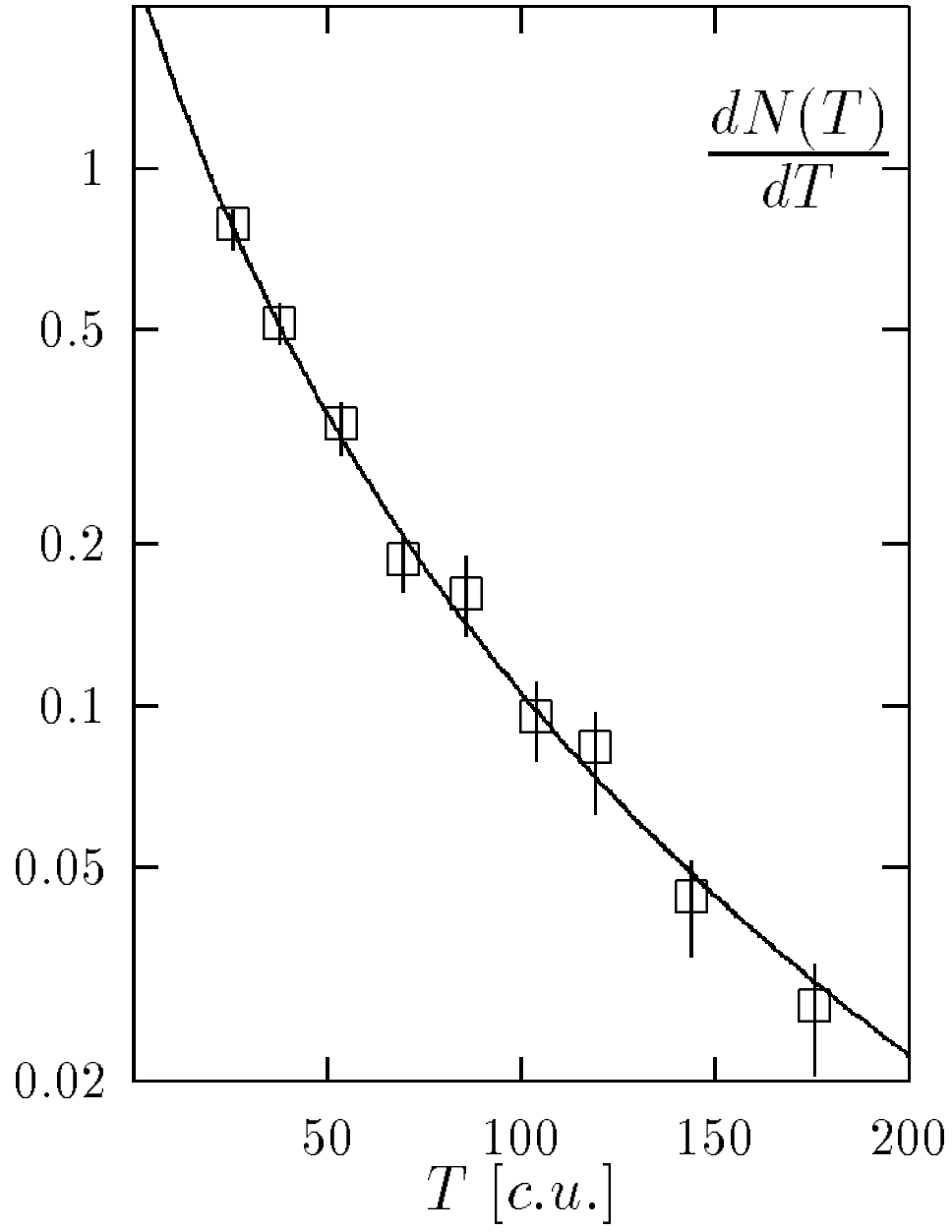


Fig. 1

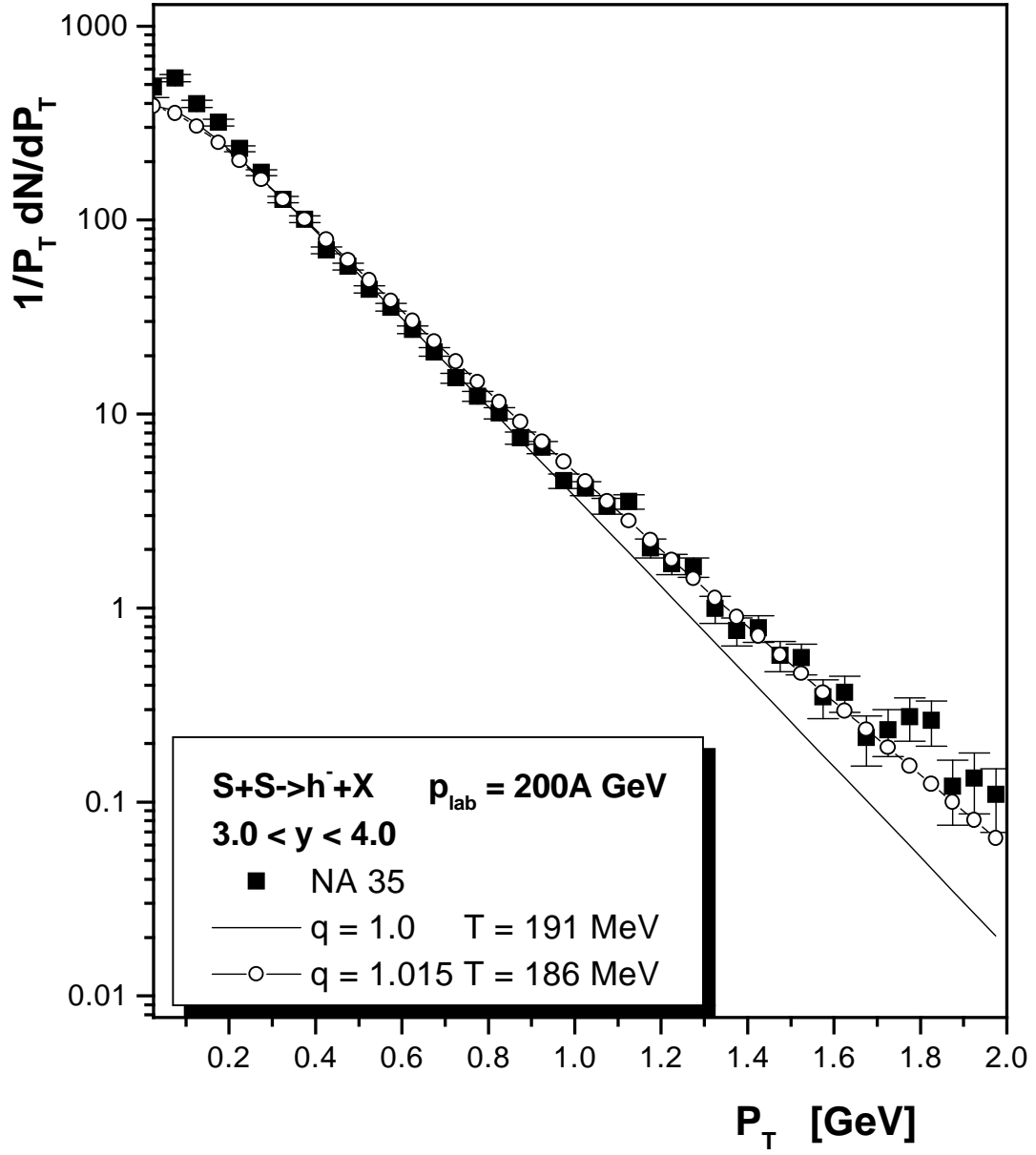


Fig.2

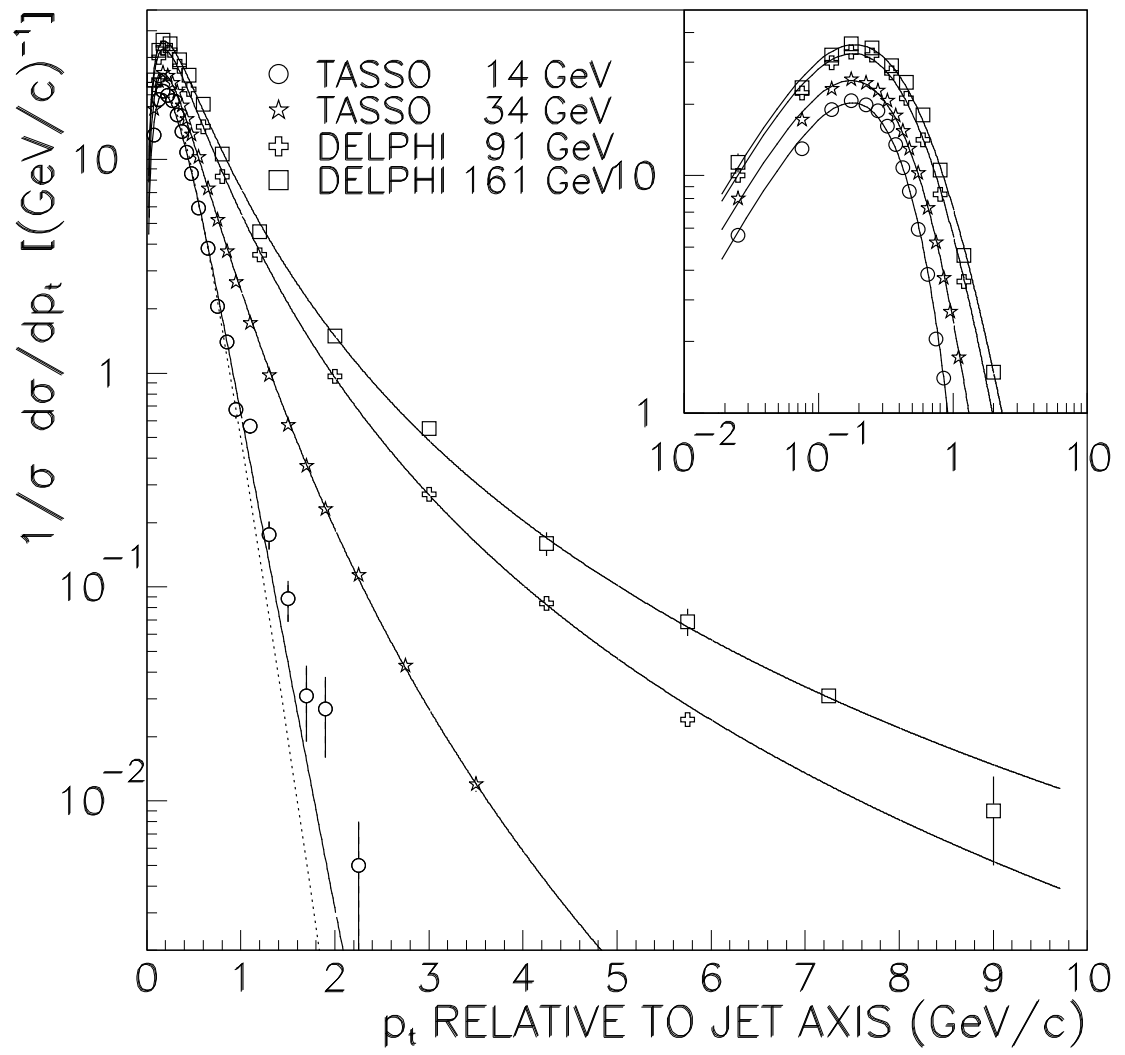


Fig.3

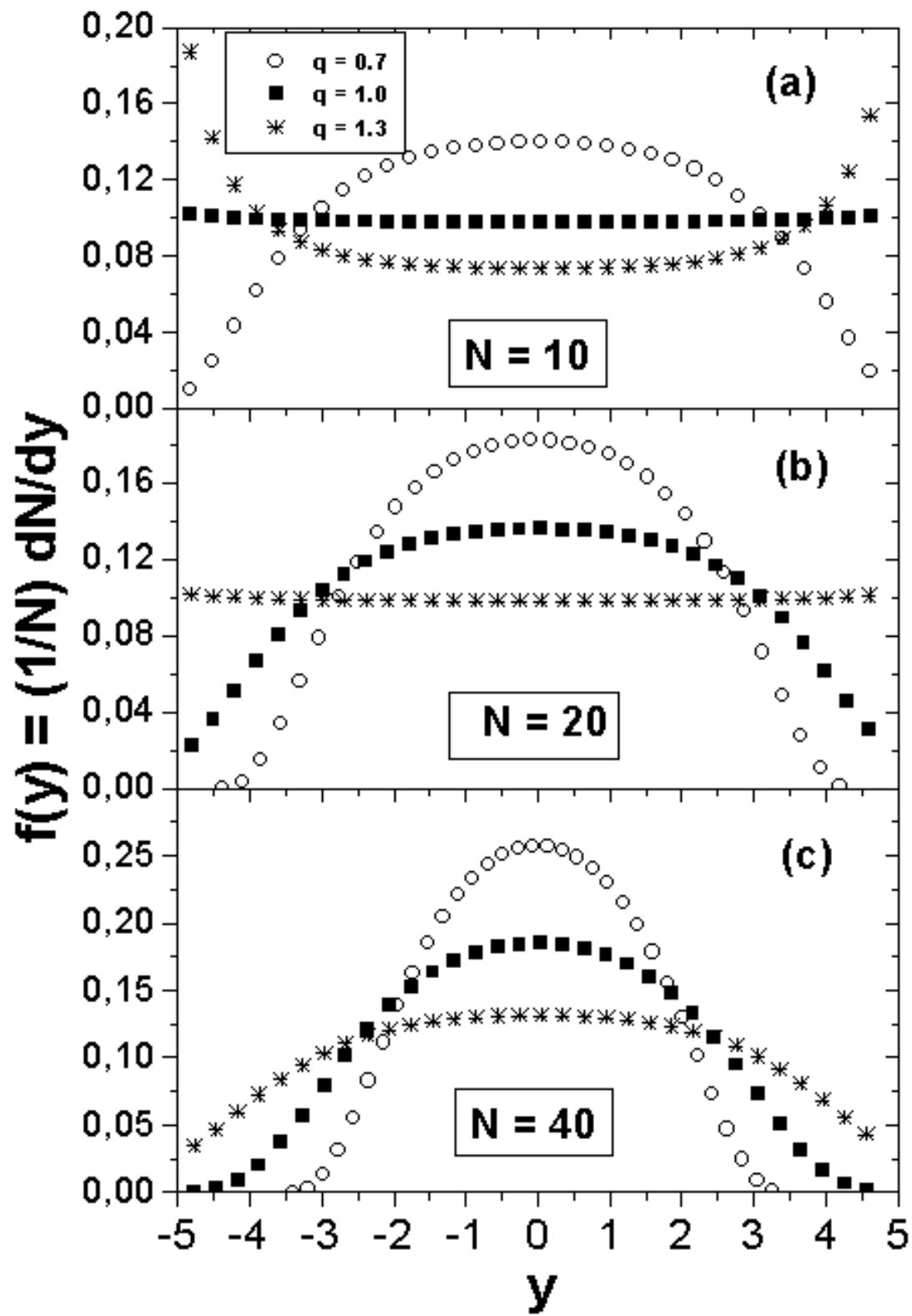


Fig.4

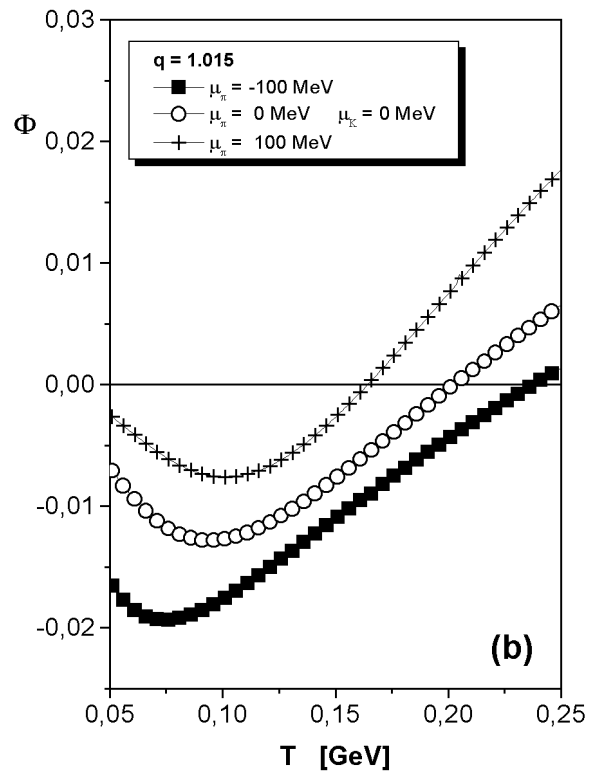
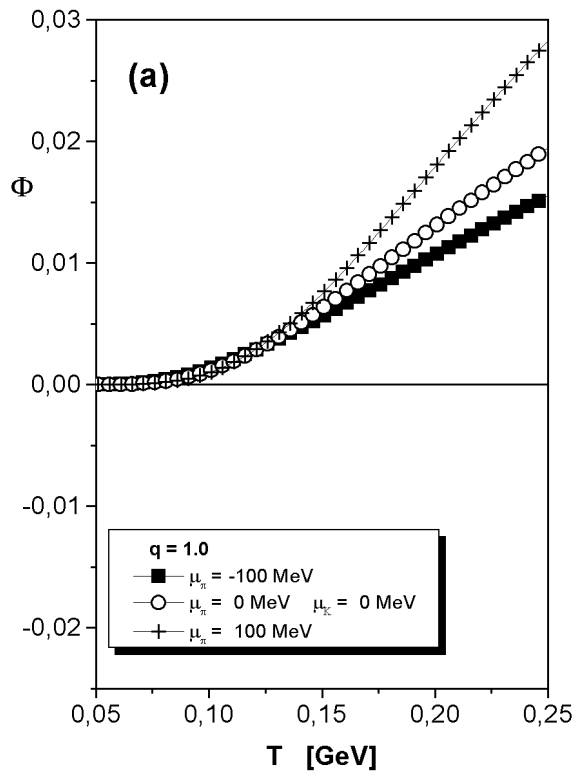


Fig.5